Lecture 27: Antiderivatives

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27.1 Equivalent conditions

Theorem 27.1. Suppose $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is continuous on D. Then f has an antiderivative F on D if and only if

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

whenever $C_1, C_2 \subset D$ have the same initial point z_1 and the same final point z_2.

Proof. Suppose f has an antiderivative F on D and let C be a smooth arc with parametrization $z(t)$, $a \le t \le b$. Let $z_1 = z(a)$ and $z_2 = z(b)$. Then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = F(z(t))\Big|_a^b = F(z_2) - F(z_1),$$

and so would be the same for any smooth arc from z_1 to z_2. If C is a contour consisting of smooth arcs C_k, with initial point z_{k-1} and final point z_k, $k = 1, 2, \ldots, n$, then

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz = \sum_{k=1}^{n} (F(z_k) - F(z_{k-1})) = F(z_n) - F(z_0),$$

a value which, again, depends only on the the initial and final points of C.

Now suppose the value of

$$\int_C f(z)dz$$

depends only on the initial and final points of C. Let $z_0 \in D$ and define

$$F(z) = \int_C f(s)ds$$

for any contour C in D with intial point z_0 and final point z. Since this value does not depend the particular contour C, we will denote the integral by

$$\int_{z_0}^{z} f(s)ds.$$

We need to show that $F'(z) = f(z)$ for any $z \in D$. Choose a γ neighborhood of z lying in D and a Δz with $0 < |\Delta z| < \gamma$. Then

$$\begin{aligned}
F(z+\Delta z) - F(z) &= \int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds \\
&= \int_{z_0}^{z} f(s)ds + \int_{z}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds \\
&= \int_{z}^{z+\Delta z} f(s)ds.
\end{aligned}$$

Now

$$\int_{z}^{z+\Delta z} ds = z\Big|_{z}^{z+\Delta z} = \Delta z,$$

and so

$$f(z) = f(z)\frac{\int_z^{z+\Delta z} ds}{\Delta z} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)ds.$$

Hence

$$\begin{aligned}
\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) &= \frac{1}{\Delta z}\left(\int_{z}^{z+\Delta z} f(s)ds - \int_{z}^{z+\Delta z} f(z)ds\right) \\
&= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} (f(s) - f(z))ds.
\end{aligned}$$

Now given $\epsilon > 0$, choose an $\alpha > 0$ such that

$$|f(s) - f(z)| < \epsilon$$

whenver $|s - z| < \alpha$. Let δ be the smaller of γ and α. Then, whenever $|\Delta z| < \delta$, we have

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|}(\epsilon |\Delta z|) = \epsilon.$$

Hence

$$\lim_{\Delta z \to 0} \left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = 0,$$

and so

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).$$

□

Theorem 27.2. Suppose $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is continuous on D. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

whenever $C_1, C_2 \subset D$ have the same initial point z_1 and the same final point z_2 if and only if

$$\int_C f(z)dz = 0$$

whenever $C \subset D$ is a closed contour.

Proof. Suppose the value of

$$\int_C f(z)dz$$

depends only on the initial and final points of C. Given a closed contour C, let z_1 and z_2 be distinct points on C. Write $C = C_1 - C_2$, where C_1 and C_2 are the two parts of C having initial point z_1 and final point z_2. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz,$$

and so

$$\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0.$$

Now suppose

$$\int_C f(z)dz = 0$$

for any closed contour $C \in D$. Let C_1 and C_2 be two contours in D, both having initial point z_1 and final point z_2. Then $C = C_1 - C_2$, and so

$$0 = \int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz.$$

Thus

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

$\square$

27.2 Examples

Example 27.1. For any contour C with initial point 0 and final point $1+i$,

$$\int_C zdz = \int_0^{1+i} zdz = \frac{1}{2}z^2\Big|_0^{1+i} = \frac{1}{2}(1+i)^2 = i.$$

Example 27.2. Since

$$F(z) = -\frac{1}{z}$$

is an antiderivative of

$$f(z) = \frac{1}{z^2}$$

on the domain $D = \{z \in \mathbb{C} : z \neq 0\}$, it follows that

$$\int_C \frac{1}{z^2}dz = 0$$

for any closed contour C in D.

Example 27.3. Let C_1 be the right half of the circle $|z| = 4$, extending from $-4i$ to $4i$. Then

$$\int_{C_1} \frac{1}{z}dz = \mathrm{Log}(z)\Big|_{-4i}^{4i} = \left(\ln(4) + i\frac{\pi}{2}\right) - \left(\ln(4) - i\frac{\pi}{2}\right) = \pi i.$$

Now let C_2 be the lefthand side of the same circle, starting at $4i$ and ending at $-4i$. Although we could not use $\mathrm{Log}(z)$ to evaluate

$$\int_{C_2} \frac{1}{z} dz,$$

we could use another branch of $\log(z)$, for example,

$$\log(z) = \ln(r) + i\theta, 0 < \theta < 2\pi.$$

Using this branch, we have

$$\int_{C_2} \frac{1}{z} dz = \log(z)\Big|_{4i}^{-4i} = \left(\ln(4) + i\frac{3\pi}{2}\right) - \left(\ln(4) + i\frac{\pi}{2}\right) = \pi i.$$

Note that $C = C_1 + C_2$ is the circle $|z| = 4$, and we have

$$\int_{C} \frac{1}{z} dz = \int_{C_1} \frac{1}{z} dz + \int_{C_2} \frac{1}{z} dz = \pi i + \pi i = 2\pi i.$$