Lecture 25:
Contour Integrals

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25.1 Contour integrals

Definition 25.1. Suppose $z(t)$, $a \le t \le b$, parametrizes a contour C and f is complex-valued function for which $f(z(t))$ is piecewise continuous on $[a, b]$. We call

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

the *contour integral* of f along C.

Example 25.1. We will evaluate

$$\int_C z^2 dz$$

where C is parametrized by $z(t) = e^{it}$, $0 \le t \le \pi$. We have

$$\begin{aligned}\int_C z^2 dz &= \int_0^\pi e^{i2t}(ie^{it})dt \\ &= i\int_0^\pi e^{3it}dt \\ &= \frac{1}{3}e^{3it}\Big|_0^\pi \\ &= \frac{1}{3}(-1-1) \\ &= -\frac{2}{3}.\end{aligned}$$

From our earlier discussion of integrals, it follows easily that if $c \in \mathbb{C}$ is a constant and f and g are complex-valued functions, then

$$\int_C cf(z)dz = c\int_C f(z)dz$$

and

$$\int_C (f(z) + g(z))dz = \int_C f(z)dz + \int_C g(z)dz.$$

Also, if C_1 and C_2 are two contours with the terminal point of C_1 the same as the initial point of C_2, and we let C denote the contour formed by C_1 and C_2 together, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

We may denote C by $C_1 + C_2$, in which case we write

$$\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

Note that if $z(t)$, $a \leq t \leq b$, parametrizes C, then

$$w(t) = z(-t), -b \leq t \leq -a,$$

parametrizes C with the opposite orientation. We denote this contour by $-C$. It follows that, using the substitution $s = -t$,

$$\begin{aligned}\int_{-C} f(z)dz &= \int_{-b}^{-a} f(w(t))w'(t)dt \\ &= -\int_{-b}^{-a} f(z(-t))z'(-t)dt \\ &= \int_b^a f(z(s)z'(s)ds \\ &= -\int_a^b f(z(s))z'(s)ds \\ &= -\int_C f(z)dz.\end{aligned}$$

Note that if C_1 and C_2 have the same terminal point, then the terminal point of C_1 is the same as the initial point of $-C_2$. Hence we may consider the contour $C_1 + (-C_1)$, which we, of course, denote $C_1 - C_2$. We have

$$\int_{C_1 - C_2} f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz.$$

25.2 Examples

Example 25.2. Let $f(x+iy) = xy + i(x+y)$ and let C be the triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$, oriented in the counterclockwise direction. To evaluate $\int_C f(z)dz$, we will write C as $C_1 + C_2 - C_3$, where C_1 has parametrization

$$z = x, 0 \le x \le 1,$$

C_2 has parametrization

$$z = 1 + iy, 0 \le y \le 1,$$

and C_3 has parametrization

$$z = x + ix, 0 \le x \le 1.$$

Then

$$\int_{C_1} f(z)dz = \int_0^1 ixdx = i\frac{1}{2},$$

$$\int_{C_2} f(z)dz = \int_0^1 (y + i(1+y))idy = -\frac{3}{2} + i\frac{1}{2},$$

and

$$\int_{C_3} f(z)dz = \int_0^1 (x^2 + i2x)(1+i)dx = \left(\frac{1}{3} + i\right)(1+i) = -\frac{2}{3} + i\frac{4}{3}.$$

Hence

$$\begin{aligned}\int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz - \int_{C_3} f(z)dz \\ &= i\frac{1}{2} - \frac{3}{2} + i\frac{1}{2} + \frac{2}{3} - i\frac{4}{3} \\ &= -\frac{5}{6} - \frac{1}{3}i.\end{aligned}$$

Note that

$$\int_{C_1+C_2} f(z)dz = i\frac{1}{2} - \frac{3}{2} + i\frac{1}{2} = -\frac{3}{2} + i \neq \int_{C_3} f(z)dz,$$

even though $C_1 + C_2$ and C_3 have the same initial and final points. Hence, although the value of a contour integral does not depend on the specific parametrization of a a given arc (see the homework), it may depend on the curve chosen to get from the intitial point to the final point.

Example 25.3. Let C, with parametrization $z(t)$, $a \leq t \leq b$, be a smooth arc and let $z_1 = z(a)$ and $z_2 = z(b)$. Then

$$\begin{aligned}
\int_C z^2 dz &= \int_a^b (z(t))^2 z'(t)dt \\
&= \left. \frac{z(t)^3}{3} \right|_a^b \\
&= \frac{z_2^3 - z_1^3}{3}.
\end{aligned}$$

Note that this means that this contour integral is independent of the particular curve starting at z_1 and ending at z_2. For example, for any curve C starting at $z_1 = 1$ and ending at $z_2 = -1$, we have

$$\int_C z^2 dz = \frac{(-1)^3 - 1^3}{3} = -\frac{2}{3}.$$

Recall that this is the result we obtained in our first example for the particular arc $z = e^{it}$, $0 \leq t \leq \pi$. This result will also hold for any contour C. Moreover, it follows that if C is a closed contour, then

$$\int_C z^2 dz = 0.$$

Example 25.4. Let C be the unit circle with parametrization $z = e^{it}$. Then

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} e^{-it}(ie^{it})dt = \int_0^{2\pi} idt = 2\pi i.$$

Does this contradict our observations in the previous example and the fact that

$$\frac{d}{dz}\log(z) = \frac{1}{z}?$$