# Lecture 22:
# Inverse Functions

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## 22.1 Inverse trigonometric functions

Given $z \in \mathbb{C}$, we would like to find $w \in \mathbb{C}$ such that $z = \sin(w)$. That is, we want to solve

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

It follows that

$$2ize^{iw} = e^{2iw} - e^0 = \left(e^{iw}\right)^2 - 1,$$

or, equivalently,

$$\left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0.$$

Using the quadratic formula, we find that

$$e^{iw} = \frac{2iz + (-4z^2 + 4)^{\frac{1}{2}}}{2} = iz + (1 - z^2)^{\frac{1}{2}}.$$

Hence

$$w = -i \log\left(iz + (1 - z^2)^{\frac{1}{2}}\right).$$

It follows that we may define the *inverse sine function*, or *arcsine function*, as

$$\sin^{-1}(z) = -i \log\left(iz + (1 - z^2)^{\frac{1}{2}}\right),$$

which we may also denote as $\arcsin(z)$. Note that this is a multi-valued function, with values depending both on the branch of the square root function and the branch of the logarithmic function chosen.

**Example 22.1.** Note that

$$\sin^{-1}(1) = -i\log(i) = -i\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right) = \frac{\pi}{2} + 2n\pi, n = 0, \pm 1, \pm 2, \ldots,$$

as we should expect.

**Example 22.2.** When $z = 0$, then $(1 - z^2)^{\frac{1}{2}} = \pm 1$. Since

$$\log(1) = 2n\pi i, n = 0, \pm 1, \pm 2, \ldots,$$

and

$$\log(-1) = i(\pi + 2n\pi) = (2n+1)\pi i, n = 0, \pm 1, \pm 2, \ldots,$$

we have

$$\sin^{-1}(0) = n\pi, n = 0, \pm 1, \pm 2, \ldots,$$

again, as we should expect.

**Example 22.3.** If $z = 2$, then $(1 - z^2)^{\frac{1}{2}} = \pm\sqrt{3}i$,

$$\log(2i + \sqrt{3}i) = \log(i(2 + \sqrt{3})) = \ln(2 + \sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots,$$

and

$$\log(2i - \sqrt{3}i) = \log(i(2 - \sqrt{3})) = \ln(2 - \sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots,$$

Now $(2 - \sqrt{3})(2 + \sqrt{3}) = 4 - 3 = 1$, so

$$2 - \sqrt{3} = \frac{1}{2 + \sqrt{3}}.$$

Hence

$$\log(2i - \sqrt{3}i) = -\ln(2 + \sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots.$$

Hence

$$\sin^{-1}(2) = \frac{\pi}{2} + 2n\pi \pm i\ln(2 + \sqrt{3}), n = 0, \pm 1, \pm 2, \ldots.$$

When specific branches of the square root and the logarithmic functions are chosen, we may differentiate $\sin^{-1}(z)$:

$$\begin{aligned}\frac{d}{dz}\sin^{-1}(z) &= \frac{d}{dz}\left(-i\log\left(iz+(1-z^2)^{\frac{1}{2}}\right)\right)\\ &= \frac{-i}{iz+(1-z^2)^{\frac{1}{2}}}\left(i+\frac{-2z}{2(1-z^2)^{\frac{1}{2}}}\right)\\ &= \frac{-i}{iz+(1-z^2)^{\frac{1}{2}}}\left(\frac{i(1-z^2)^{\frac{1}{2}}-z}{(1-z^2)^{\frac{1}{2}}}\right)\\ &= \frac{1}{(1-z^2)^{\frac{1}{2}}}.\end{aligned}$$

Note that the result depends on the branch of the square root function chosen, but not on the particular branch of the logarithmic function. Also, note that $\sin^{-1}(z)$ has singular points at $\pm 1$.

We may define, in an analogous manner, inverse functions for the remaining circular trigonometric functions. The most important of these are

$$\cos^{-1}(z) = -i\log\left(z+i(1-z^2)^{\frac{1}{2}}\right)$$

and

$$\tan^{-1}(z) = \frac{i}{2}\log\frac{i+z}{i-z},$$

with derivatives

$$\frac{d}{dz}\cos^{-1}(z) = -\frac{1}{(1-z^2)^{\frac{1}{2}}}$$

and

$$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}.$$

## 22.2 Inverse hyperbolic functions

Given $z \in \mathbb{C}$, we would like to find $w \in \mathbb{C}$ such that $z = \tanh(w)$. That is, we want to solve

$$z = \frac{e^w - e^{-w}}{e^w + e^{-w}}.$$

It follows that

$$ze^{2w} + z = e^{2w} - 1,$$

or, equivalently,

$$e^{2w} = \frac{1+z}{1-z}.$$

Hence

$$w = \frac{1}{2} \log \frac{1+z}{1-z}.$$

Thus we define the *inverse hyperbolic tangent function* by

$$\tanh^{-1}(z) = \frac{1}{2} \log \frac{1+z}{1-z}.$$

We find the other inverse hyperbolic trigonometric functions in a similar manner. The most important of these are

$$\sinh^{-1}(z) = \log \left( z + (z^2+1)^{\frac{1}{2}} \right)$$

and

$$\cosh^{-1}(z) = \log \left( z + (z^2-1)^{\frac{1}{2}} \right).$$

The derivatives are

$$\frac{d}{dz} \sinh^{-1}(z) = \frac{1}{(z^2+1)^{\frac{1}{2}}},$$

$$\frac{d}{dz} \cosh^{-1}(z) = \frac{1}{(z^2-1)^{\frac{1}{2}}},$$

and

$$\frac{d}{dz} \tanh^{-1}(z) = \frac{1}{1-z^2}.$$