Lecture 21:
Hyperbolic Functions

Dan Sloughter
Furman University
Mathematics 39

April 8, 2004

21.1 Hyperbolic sine and cosine

Definition 21.1. For any $z \in \mathbb{C}$, we define the *hyperbolic sine function* by

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

and the *hyperbolic cosine function* by

$$\cosh(z) = \frac{e^z + e^{-z}}{2}.$$

Proposition 21.1. For any $z \in \mathbb{C}$,

$$\frac{d}{dz}\sinh(z) = \cosh(z)$$

and

$$\frac{d}{dz}\cosh(z) = \sinh(z).$$

Proof. We have

$$\frac{d}{dz}\sinh(z) = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^z + e^{-z}}{2} = \cosh(z)$$

and

$$\frac{d}{dz}\cosh(z) = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^z - e^{-z}}{2} = \sinh(z).$$

□

Note that

$$\cos(z) = \cosh(iz) \text{ and } \sin(z) = -i\sinh(iz)$$

and

$$\cosh(z) = \cos(iz) \text{ and } \sinh(z) = -i\sin(iz).$$

It follows that

$$\sinh(-z) = -i\sin(-iz) = i\sin(iz) = -\sinh(z),$$

$$\cosh(-z) = \cos(-iz) = \cos(iz) = \cosh(z),$$

$$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) - (-i\sin(iz))^2 = \cos^2(iz) + \sin^2(iz) = 1,$$

$$\begin{aligned}\sinh(z_1 + z_2) &= -i\sin(i(z_1 + z_2)\\ &= -i\sin(iz_1)\cos(iz_2) - i\cos(iz_1)\sin(iz_2)\\ &= \sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2),\end{aligned}$$

and

$$\begin{aligned}\cosh(z_1 + z_2) &= \cos(i(z_1 + z_2))\\ &= \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)\\ &= \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2).\end{aligned}$$

Moreover, if $z = x + iy$, then $iz = -y + ix$, and we have

$$\begin{aligned}\sinh(z) &= -i\sin(iz)\\ &= -i(\sin(-y)\cos(ix) + \cos(-y)\sin(ix)\\ &= \sinh(x)\cos(y) + \cosh(x)\sin(y),\end{aligned}$$

$$\begin{aligned}\cosh(z) &= \cos(iz)\\ &= \cos(-y)\cos(ix) - \sin(-y)\sin(ix)\\ &= \cosh(x)\cos(y) + i\sinh(x)\sin(y),\end{aligned}$$

$$|\sinh(z)|^2 = |-i\sin(iz)|^2 = \sin^2(-y) + \sinh^2(x) = \sinh^2(x) + \sin^2(y)$$

and

$$|\cosh(z)|^2 = |\cosh(iz)|^2 = \cos^2(-y) + \sinh^2(x) = \sinh^2(x) + \cos^2(y).$$

Also note that it follows that both $\sinh(z)$ and $\cosh(z)$ are periodic with period $2\pi i$, that $\sinh(z) = 0$ if and only if $z = n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$, and $\cosh(z) = 0$ if and only if $z = i\left(\frac{\pi}{2} + n\pi\right)$, $n = 0, \pm 1, \pm 2, \ldots$.

21.2 The other hyperbolic functions

In analogy with the the circular trigonometric functions, we also define

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)},$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)},$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)},$$

and

$$\operatorname{csch}(z) = \frac{1}{\operatorname{sech}(z)}.$$

These are all analytic in their domains of definition. One may show, as in the real-variable case, that

$$\frac{d}{dz}\tanh(z) = \operatorname{sech}^2(z),$$

$$\frac{d}{dz}\coth(z) = -\operatorname{csch}^2(z),$$

$$\frac{d}{dz}\operatorname{sech}(z) = -\operatorname{sech}(z)\tanh(z),$$

and

$$\frac{d}{dz}\operatorname{csch}(z) = -\operatorname{csch}(z)\coth(z).$$