Lecture 2:
Algebra of Complex Numbers

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2.1 Some algebraic properties

It is easy to show that the addition of complex numbers is *commutative* and *associative*; that is, for any complex numbers z_1, z_2, and z_3,

$$z_1 + z_2 = z_2 + z_1$$

and

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

For multiplication, note that if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2) = (x_2 x_1 - y_2 y_1, y_2 x_1 + x_2 y_1) = z_2 z_1.$$

Hence multiplication is commutative. One may show as well that multiplication is associative, that is,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3),$$

and that multiplication distributes over addition:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Addition and multiplication have unique *identities*, namely, $0 = (0, 0)$ and $1 = (1, 0)$; that is, for every complex number z,

$$0 + z = z \text{ and } 1 \cdot z = z.$$

Moreover, each $z = (x, y))$ has a unique *additive inverse*: if $-z = (-x, -y)$ then

$$z + (-z) = 0.$$

Note that this enables us to define subtraction:

$$z_1 - z_2 = z_1 + (-z_2).$$

That is, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2).$$

To find a multiplicative inverse, note that, given $z = (x, y)$, $z \neq 0$, we need to find $w = (u, v)$ such that $zw = (1, 0)$, that is, such that

$$xu - yv = 1 \text{ and } yu + xv = 0.$$

Multiplying the first equation by x, the second by y, and adding, we have

$$x^2u + y^2u = x.$$

Since $x^2 + y^2 \neq 0$, we have

$$u = \frac{x}{x^2 + y^2}.$$

Multiplying the first equation by y, the second by x, and subtracting, we have

$$-y^2v - x^2v = y,$$

from which it follows that

$$v = -\frac{y}{x^2 + y^2}.$$

Hence the unique multiplicative inverse of $z = x + iy$ is

$$z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

Example 2.1. If $z = 1 + 2i$, then

$$z^{-1} = \frac{1}{5} - \frac{2}{5}i,$$

which we may check by noting that

$$(1 + 2i)\left(\frac{1}{5} - \frac{2}{5}i\right) = \frac{1}{5} + \frac{4}{5} + \left(-\frac{2}{5} + \frac{2}{5}\right)i = 1.$$

Now that we know that multiplicative inverses exist, we may proceed as follows:

$$z^{-1} = \frac{1}{1+2i} = \frac{1}{1+2i}\frac{1-2i}{1-2i} = \frac{1-2i}{1+4} = \frac{1}{5} - \frac{2}{5}i.$$

Now suppose $z_1 \neq 0$ and $z_1 z_2 = 0$. Then

$$z_2 = (z_1^{-1} z_1) z_2 = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0.$$

That is, if $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.

We may now define division: if $z_2 \neq 0$, we define

$$\frac{z_1}{z_2} = z_1 z_2^{-1}.$$

Although one may write out a formula for division, in practice it is usually preferable to follow the next example.

Example 2.2. We have

$$\frac{3+4i}{1-2i} = \frac{3+4i}{1-2i}\frac{1+2i}{1+2i} = \frac{(3-8)+(6+4)i}{1+4} = -\frac{5}{5} + \frac{10}{5}i = -1+2i.$$

In the language of algebra, we have shown that $\mathbb{C}$ is a *field*, and we may work with complex numbers algebraically the same way we work with real numbers.