Lecture 17: Exponentials and Logs

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17.1 The exponential function

Recall that for any $z = x + iy \in \mathbb{C}$, we have defined

$$\exp(z) = e^z = e^x e^{iy}.$$

Note that, in an exception to our earlier convention, we will interpret

$$e^{\frac{1}{n}} = \sqrt[n]{e},$$

not as the set of nth roots of e.

Proposition 17.1. For any $z \in \mathbb{C}$, $e^z \neq 0$.

Proof. Note that, if $z = x + iy$,

$$|e^z| = |e^x||e^{iy}| = e^x \neq 0$$

for all $x \in \mathbb{R}$. □

Recall, however, that e^z may assume negative values. For example,

$$e^{i\pi} = -1.$$

Proposition 17.2. For any $z_1, z_2 \in \mathbb{C}$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$ and

$$e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}.$$

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} e^{z_1+z_2} &= e^{(x_1+x_2)+i(y_2+y_2)} \\ &= e^{x_1+x_2}e^{i(y_1+y_2)} \\ &= e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} \\ &= e^{x_1+iy_1}e^{x_2+iy_2} \\ &= e^{z_1}e^{z_2}. \end{aligned}$$

It now follows that

$$e^{z_1-z_2}e^{z_2} = e^{z_1-z_2+z_2} = e^{z_1},$$

and so

$$e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}.$$

□

Proposition 17.3. e^z is periodic with period $2\pi i$.

Proof. For any $z \in \mathbb{C}$,

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z.$$

□

Example 17.1. The solutions of the equation $e^z = -1$ are

$$z = \pi i + 2n\pi i = (2n+1)\pi i, n = 0, \pm 1, \pm 2, \ldots.$$

Is this $\log(-1)$?

17.2 The logarithmic function

Given any $z \in \mathbb{C}$, $z \neq 0$, we want to find w such that

$$e^w = z.$$

Now if $z = re^{i\Theta}$, with $r > 0$ and $-\pi < \Theta \leq \pi$, and $w = u + iv$, then we want

$$e^u e^{iv} = re^{i\Theta}$$

from which it follows that

$$e^u = r \text{ and } v = \Theta + 2n\pi, n = 0, \pm 1, \pm 2, \ldots.$$

It follows that $u = \ln(r)$, and so

$$w = \ln(r) + i(\Theta + 2n\pi), n = 0, \pm 1, \pm 2, \ldots.$$

Definition 17.1. Given $z \in \mathbb{C}$, $z \neq 0$, we call

$$\log(z) = \ln(|z|) + i(\mathrm{Arg}\,(z) + 2n\pi), n = 0, \pm 1, \pm 2, \ldots$$

the (multi-valued) *logarithmic function* of z. We call

$$\mathrm{Log}\,(z) = \ln(|z|) + i\mathrm{Arg}\,(z)$$

the *principal value* of $\log(z)$.

Example 17.2. If $z = 1 - i$, then

$$\mathrm{Log}\,(z) = \ln(\sqrt{2}) - \frac{\pi}{4}i = \frac{1}{2}\ln(2) - \frac{\pi}{4}i$$

and

$$\log(z) = \frac{1}{2}\ln(2) + i\left(-\frac{\pi}{4} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots.$$

Note that $e^{\log(z)} = z$, but, if $z = x + iy$,

$$\log(e^z) = \ln(|e^z|) + i\arg(e^z) = \ln(e^x) + i(y + 2n\pi) = z + 2n\pi i, n = 0, \pm 1, \pm 2, \ldots.$$

Example 17.3. If $z = 4\pi i$, then

$$\log(e^z) = i(4\pi + 2n\pi), n = 0, \pm 1, \pm 2, \ldots,$$

and

$$\mathrm{Log}\,(e^z) = \mathrm{Log}\,(1) = 0.$$