Lecture 16: Harmonic Functions

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Mathematics 39

March 31, 2004

16.1 Harmonic functions

Suppose f is analytic in a domain D,

$$f(x+iy) = u(x,y) + iv(x,y),$$

and u and v have continuous partial derivatives of all orders. From the Cauchy-Riemann equations we know that

$$u_x(x,y) = v_y(x,y) \text{ and } u_y(x,y) = -v_x(x,y)$$

for all $x + iy \in D$. Differentiating with respect to x, we have

$$u_{xx}(x,y) = v_{yx}(x,y) \text{ and } u_{yx}(x,y) = -v_{xx}(x,y);$$

differentiating with respect to y, we have

$$u_{xy}(x,y) = v_{yy}(x,y) \text{ and } u_{yy}(x,y) = -v_{xy}(x,y).$$

Hence

$$u_{xx}(x,y) + u_{yy}(x,y) = v_{yx}(x,y) - v_{xy}(x,y) = v_{xy}(x,y) - v_{xy}(x,y) = 0$$

for all $x + iy \in D$ and

$$v_{xx}(x,y) + v_{yy}(x,y) = -u_{yx}(x,y) + u_{xy}(x,y) = -u_{xy}(x,y) + u_{xy}(x,y) = 0$$

for all $x + iy \in D$.

Definition 16.1. Suppose $H : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives on a domain D. We say H is *harmonic* in D if for all $(x, y) \in D$,

$$H_{xx}(x, y) + H_{yy}(x, y) = 0.$$

Harmonic functions arise frequently in applications, such as in the study of heat distributions and electrostatic potentials.

Theorem 16.1. If f is analytic in a domain D and

$$f(x + iy) = u(x, y) + iv(x, y),$$

then u and v are harmonic in D.

Proof. The result follows from the discussion above combined with a result we will prove later: if f is analytic at $z_0 = x_0 + iy_0$, then u and v have continuous partial derivatives of all orders at (x_0, y_0). □

Example 16.1. We know that $f(z) = e^z$ is entire. Since

$$f(x + iy) = e^x \cos(y) + ie^x \sin(y),$$

it follows that $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$ are both harmonic in $\mathbb{C}$ (which is also easily checked directly).

Example 16.2. We know that

$$f(z) = \frac{1}{z^2}$$

is analytic in $\{z \in \mathbb{C} : z \neq 0\}$. Now

$$\frac{1}{z^2} = \frac{1}{z^2}\frac{\bar{z}^2}{\bar{z}^2} = \frac{x^2 - y^2 - 2xyi}{(x^2 + y^2)^2},$$

so

$$u(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

and

$$v(x, y) = -\frac{2xy}{(x^2 + y^2)^2}$$

are harmonic in $\{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$.

Definition 16.2. If u and v are harmonic in a domain D and satisfy the Cauchy-Riemann equations, then we say v is a *harmonic conjugate* of u.

Example 16.3. It is easy to check that the function

$$u(x, y) = x^3 - 3xy^2$$

is harmonic. To find a harmonic conjugate v of u, we must have

$$u_x(x, y) = v_y(x, y)$$

and

$$u_y(x, y) = -v_x(x, y).$$

From the first we have

$$v_y(x, y) = 3x^2 - 3y^2,$$

from which it follows that

$$v(x, y) = 3x^2y - y^3 + \varphi(x)$$

for some function φ of x. It now follows from the second equation that

$$-6xy = -v_x(x, y) = -(6xy + \varphi'(x)),$$

and so $\varphi'(x) = 0$. Hence for any real number c, the function

$$v(x, y) = 3x^2y - y^3 + c$$

is a harmonic conjugate of u.