Lecture 15:
Analytic Functions

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15.1 Analytic functions

Definition 15.1. We say a function f is *analytic* (or *regular* or *holomorphic*) in an open set U if f is differentiable at each point $z \in U$. We say f is *analytic* in a set S (not necessarily open) if f is analytic in an open set containing S. We say f is *analytic* at a point z_0 if it is analytic in some neighborhood of z_0.

Definition 15.2. We say a function f is *entire* if it is analytic in $\mathbb{C}$.

Example 15.1. Every polynomial is an entire function.

Example 15.2. The function

$$f(z) = \frac{1}{z}$$

is analytic in $U = \{z \in \mathbb{C} : z \neq 0\}$.

Example 15.3. There are no points at which the function $f(z) = |z|^2$ is analytic.

Definition 15.3. If f is not analytic at z_0 but is analytic at some point in every neighborhood of z_0, then we call z_0 a *singular point* of f.

Example 15.4. $z = 0$ is a singular point of

$$f(z) = \frac{1}{z}.$$

Example 15.5. $f(z) = |z|^2$ has no singular points.

Example 15.6. The function

$$f(z) = \frac{1}{z^2 + 1}$$

is analytic in $U = \{z \in \mathbb{C} : z \neq \pm i\}$.

Theorem 15.1. If $f'(z) = 0$ for every z in a domain D, then there exists a constant $a \in \mathbb{C}$ such that $f(z) = a$ for all $z \in \mathbb{C}$.

Proof. If

$$f(x + iy) = u(x, y) + iv(x, y),$$

then it follows that $u_x(x, y) = 0$ and $v_x(x, y) = 0$ for all $x + iy \in D$. From the Cauchy-Riemann equations, it follows that $u_y(x, y) = 0$ and $v_y(x, y) = 0$ for all $x + iy \in D$ as well. It now follows that, from results in multi-variable calculus, that $u(x, y) = a$ and $v(x, y) = b$ for some constants $a, b \in \mathbb{R}$ and for all $x + iy \in D$. $\square$

Example 15.7. Suppose

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

and

$$g(z) = \overline{f(z)} = u(x, y) - iv(x, y)$$

are both analytic on a domain D. From the Cauchy-Riemann equations, the analyticity of f implies

$$u_x(x, y) = v_y(x, y) \text{ and } u_y(x, y) = -v_x(x, y)$$

for all $x + iy \in D$ and the analyticity of g implies

$$u_x(x, y) = -v_y(x, y) \text{ and } u_y(x, y) = v_x(x, y)$$

for all $x + iy \in D$. It follows that $2u_x(x, y) = 0$ and $2u_y(x, y) = 0$ for all $x + iy \in D$. Hence

$$f'(z) = u_x(x, y) + iv_x(x, y) = 0$$

for all $z \in D$, and so f is constant on D.