Lecture 11: Continuity

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11.1 Continuous functions

Definition 11.1. Suppose $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$. We say f is *continuous* at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If S is a region, we say f is *continuous* on S if f is continuous at each point of S.

Note that f is continuous at z_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon$$

whenever

$$|z - z_0| < \delta.$$

Proposition 11.1. If f and g are both continuous at z_0, then the functions

$$h(z) = f(z) + g(z)$$

and

$$k(z) = f(z)g(z)$$

are continuous at z_0. Moreover, if $g(z_0) \neq 0$, then the function

$$r(z) = \frac{f(z)}{g(z)}$$

is continuous at z_0.

Proof. These results follow immediately from our results about limits of sums, products, and quotients. □

Example 11.1. If $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, then P is continuous on $\mathbb{C}$.

Example 11.2. If R is a rational function and z_0 is a point in the domain of R, then R is continuous at z_0.

Proposition 11.2. Suppose $f(x + iy) = u(x, y) + iv(x, y)$. Then f is continuous at $z_0 = x_0 + iy_0$ if and only if u and v are both continuous at (x_0, y_0).

Proof. This, again, is a consequence of the corresponding result about limits. □

Example 11.3. The function

$$f(x + iy) = (x^2 - 2xy) + i \sin(x + y)$$

is continuous on $\mathbb{C}$.

Proposition 11.3. If f is continuous at z_0 and g is continuous at $w_0 = f(z_0)$, then $g \circ f$ is continuous at z_0.

Proof. Given $\epsilon > 0$, we need to find $\delta > 0$ such that

$$|g(f(z)) - g(f(z_0))| < \epsilon$$

whenever

$$|z - z_0| < \delta.$$

Since g is continuous at w_0, we may choose $\delta_1 > 0$ so that

$$|g(w) - g(w_0)| < \epsilon$$

whenever

$$|w - w_0| < \delta_1.$$

Since f is continuous at z_0, we may choose $\delta > 0$ so that

$$|f(z) - f(z_0)| < \delta_1$$

whenever

$$|z - z_0| < \delta.$$

Since $f(z_0) = w_0$, it now follows that

$$|g(f(z)) - g(f(z_0))| < \epsilon$$

whenever

$$|z - z_0| < \delta,$$

and so $\lim_{z \to z_0} g(f(z)) = g(f(z_0))$. Hence $g \circ f$ is continuous at z_0. □

Example 11.4. The function

$$f(z) = |z|$$

is continuous on $\mathbb{C}$ (f is the composition of the function $g(x+iy) = x^2 + y^2$ with the function $h(x) = \sqrt{x}$).

Proposition 11.4. If f is continuous at z_0 with $f(z_0) \neq 0$, then there exists a neighborhood U of z_0 for which $f(z) \neq 0$ for all $z \in U$.

Proof. Let

$$\epsilon = \frac{|f(z_0)|}{2}.$$

Then there exists $\delta > 0$ such that, if U is the δ neighborhood of z_0,

$$|f(z) - f(z_0)| < \epsilon$$

whenever $z \in U$. It follows that if $z \in U$,

$$|f(z)| = |(f(z) - f(z_0)) + f(z_0)| \geq ||f(z_0)| - |f(z) - f(z_0)|| > \frac{|f(z_0)|}{2}.$$

Hence $f(z) \neq 0$ for all $z \in U$. □