Lecture 10:
The Point at Infinity

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10.1 The point at infinity

Let S be the unit sphere in $\mathbb{R}^3$ with center at $(0, 0, 0)$ and let $N = (0, 0, 1)$, the *north pole* of S. If P is any point on S other than N, then the line through N and P intersects the xy-plane, considered as the complex plane $\mathbb{C}$, in a unique point z. Note that if $P = (u, v, w)$ with $w > 0$, then z lies outside the unit circle C centered at the origin; if $w = 0$, then z lies on C; and if $w < 0$, then z lies inside C, with $z = 0$ if $P = (0, 0, -1)$. In this way we create a one-to-one correspondence between points $P \neq N$ on the sphere and the complex plane. We extend this one-to-one correspondence by letting N be the *point at infinity*, thus creating a one-to-one correspondence between S and what we will call the *extended complex plane*.

Note that a neighborhood of N on S corresponds to a set in $\mathbb{C}$ of the form

$$\left\{ z \in \mathbb{C} : |z| > \frac{1}{\epsilon} \right\}$$

for some $\epsilon > 0$. For this reason, we call such a set an ϵ *neighborhood* of ∞. Using such neighborhoods in addition to those defined previously, we may extend the definition of

$$\lim_{z \to z_0} f(z) = w_0$$

to include the cases where one or both of z_0 and w_0 is the point at infinity.

Proposition 10.1. Let $z_0 \in \mathbb{C}$ and $w_0 \in \mathbb{C}$. Then

$$\lim_{z \to z_0} f(z) = \infty \text{ if and only if } \lim_{z \to z_0} \frac{1}{f(z)} = 0;$$

$$\lim_{z \to \infty} f(z) = w_0 \text{ if and only if } \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0;$$

and

$$\lim_{z \to \infty} f(z) = \infty \text{ if and only if } \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$$

Example 10.1. To evaluate

$$\lim_{z \to i} \frac{z+3}{z-i},$$

we first evaluate

$$\lim_{z \to i} \frac{z-i}{z+3} = \frac{0}{3+i} = 0.$$

Thus

$$\lim_{z \to i} \frac{z+3}{z-i} = \infty.$$

Example 10.2. To evaluate

$$\lim_{z \to \infty} \frac{iz+4}{4z+i},$$

we first evaluate

$$\lim_{z \to 0} \frac{\frac{i}{z}+4}{\frac{4}{z}+i} = \lim_{z \to 0} \frac{i+4z}{4+iz} = \frac{1}{4}i.$$

Hence

$$\lim_{z \to \infty} \frac{iz+4}{4z+i} = \frac{1}{4}i.$$

Example 10.3. To evaluate

$$\lim_{z \to \infty} \frac{5z^2+1}{z+3},$$

we first evaluate

$$\lim_{z \to 0} \frac{\frac{1}{z}+3}{\frac{5}{z^2}+1} = \lim_{z \to 0} \frac{z+3z^2}{5+z^2} = \frac{0}{5} = 0.$$

Hence

$$\lim_{z \to \infty} \frac{5z^2+1}{z+3} = \infty.$$