

Lecture 1: Complex Numbers

Dan Sloughter
Furman University
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1.1 Basic definitions

Definition 1.1. Given $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, we define the *sum* of z_1 and z_2 by

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

and the *product* of z_1 and z_2 by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2).$$

With these definitions of sum and product, we call

$$\mathbb{C} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$$

the *complex plane* and we call $z \in \mathbb{C}$ a *complex number*. If $z = (x, y) \in \mathbb{C}$, we call x the *real part* and y the *imaginary part* of z , and we write

$$\operatorname{Re} z = x \text{ and } \operatorname{Im} z = y.$$

We call the set $\{(x, 0) : x \in \mathbb{R}\}$ the *real axis* and the set $\{(0, y) : y \in \mathbb{R}\}$ the *imaginary axis*. We say numbers on the imaginary axis are *pure imaginary numbers*.

Note that if $z_1 = (x_1, 0)$ and $z_2 = (x_2, 0)$ are numbers on the real axis, then

$$z_1 + z_2 = (x_1 + x_2, 0)$$

and

$$z_1 z_2 = (x_1 x_2, 0).$$

That is, addition and multiplication behave just as they do for real numbers. Hence we may think of numbers on the real axis as real numbers, and the complex numbers as an extension of the real numbers. Moreover, we may write x for $(x, 0)$.

We also have

$$(0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0).$$

If we let $i = (0, 1)$, then, for any $y \in \mathbb{R}$,

$$i(y, 0) = (0, 1)(y, 0) = (0 - 0, 0 + y) = (0, y).$$

Hence for any $(x, y) \in \mathbb{C}$,

$$(x, y) = (x, 0) + (0, y) = (x, 0) + i(y, 0) = x + iy.$$

It follows that

$$i^2 = (-1, 0) = -1,$$

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$(x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2).$$

Example 1.1. If $z = 4 + 3i$ and $w = 2 - 6i$, then

$$z + w = 6 - 3i$$

and

$$zw = (8 + 18) + (6 - 24)i = 26 - 18i.$$